Algebra 2: Remote Learning Packet

Directions:
1. Sign up for the Algebra 2 Google Classroom (class code: fvye55k; you should also have a link in your KIPP email!)

2. There are 10 total sections in this packet (one section per day). Examples are given to remind you how to complete the problems. All problems packet must be completed in pencil on a separate sheet of notebook paper.

3. Answers to the problems in this packet will be posted on Monday, 3/30/2020. Use the answers to check your work. Any errors found must be corrected.

If you need additional resources, you should use...
- The Algebra 2 website (sites.google.com/a/kippnashville.org/mstil) referencing units 3 (polynomials), 4 (radicals), 5 (inverse/composite functions), and 6 (exponential functions).
- Use videos on YouTube and Khan Academy (try searching for the title of the lesson!).
- Reach out to Mr. Shannon or Ms. Tillotson:
  - Mr. Shannon: rshannon@kippnashville.org
    (615) 601-2313
  - Ms. Tillotson: jtillotson@kippnashville.org
    (203) 535-4644
- Attend virtual office hours for one-on-one real-time support
I. Characteristics & transformations of polynomial functions

**Key Concept**

**Polynomial in One Variable**

- **Words**
  A polynomial of degree \( n \) in one variable \( x \) is an expression of the form \( a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0 \), where the coefficients \( a_n, a_{n-1}, a_2, \ldots, a_1, a_0 \) represent real numbers, \( a_0 \) is not zero, and \( n \) represents a nonnegative integer.

- **Examples**
  \[ 3x^3 + 2x^2 - 5x^2 + x^2 + 1 \]
  \[ n = 5, \ a_0 = 3, \ a_1 = 2, \ a_2 = -5, \ a_3 = 1, \ a_4 = 0, \text{ and } a_5 = 1 \]

The **degree of a polynomial** in one variable is the greatest exponent of its variable. The **leading coefficient** is the coefficient of the term with the highest degree.

<table>
<thead>
<tr>
<th>Polynomial</th>
<th>Expression</th>
<th>Degree</th>
<th>Leading Coefficient</th>
</tr>
</thead>
<tbody>
<tr>
<td>Constant</td>
<td>9</td>
<td>0</td>
<td>9</td>
</tr>
<tr>
<td>Linear</td>
<td>( x - 2 )</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Quadratic</td>
<td>( 3x^2 + 4x - 5 )</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>Cubic</td>
<td>( 4x^3 - 6 )</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>General</td>
<td>( a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0 )</td>
<td>( n )</td>
<td>( a_0 )</td>
</tr>
</tbody>
</table>

**Example 1** Find Degree and Leading Coefficient

State the degree and leading coefficient of each polynomial in one variable. If it is not a polynomial in one variable, explain why.

a. \( 7x^4 + 5x^2 + x - 9 \)
   This is a polynomial in one variable. The degree is 4, and the leading coefficient is 7.

b. \( 8x^2 + 3xy - 2y^2 \)
   This is not a polynomial in one variable. It contains two variables, \( x \) and \( y \).

c. \( 7x^6 - 4x^3 + \frac{1}{x} \)
   This is not a polynomial. The term \( \frac{1}{x} \) cannot be written in the form \( x^n \), where \( n \) is a nonnegative integer.

d. \( \frac{1}{2} x^3 + 2x^3 - x^5 \)
   Rewrite the expression so the powers of \( x \) are in decreasing order.
   \[ -x^5 + 2x^3 + \frac{1}{2} x^2 \]
   This is a polynomial in one variable with degree of 5 and leading coefficient of \(-1\).

---

**Study Tip**

**Power Function**

A common type of function is a power function, which has an equation in the form \( f(x) = ax^b \), where \( a \) and \( b \) are real numbers. When \( b \) is a positive integer, \( f(x) = ax^b \) is a polynomial function.

---

**Key Concept**

**Definition of a Polynomial Function**

- **Words**
  A polynomial function of degree \( n \) can be described by an equation of the form \( P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0 \), where the coefficients \( a_n, a_{n-1}, a_2, \ldots, a_1, a_0 \) represent real numbers, \( a_0 \) is not zero, and \( n \) represents a nonnegative integer.

- **Examples**
  \[ f(x) = 4x^2 - 3x + 2 \]
  \[ n = 2, \ a_0 = 4, \ a_1 = -3, \ a_2 = 2 \]
Comprehension questions: part 1 (characteristics of polynomial functions)

Answer each of the following questions on a separate sheet of paper.

1. Determine whether each expression below represents a polynomial in one variable. If so, state the degree and leading coefficient. If not, explain why.
   a) \( 5x^6 - 8x^2 \)
   b) \( 2b + 4b^3 - 3b^5 - 7 \)
   c) \( c^2 - \frac{5}{c} + 18 \)

2. For each function below, explain how the graph of the function differs from the graph of the parent function, \( f(x) = x^3 \). Then graph the function without using a calculator.
   a) \( g(x) = (x + 5)^3 + 5 \)
   b) \( h(x) = x^3 + 5 \)
   c) \( j(x) = -\frac{1}{3}(x - 1)^3 \)

3. A certain cubic function is given by \( m(x) = -2x^3 - 4 \). Without graphing, identify the three differences this function displays relative to the parent function.
II. Identifying factors of polynomial functions

Key Concept

The binomial \( x - a \) is a factor of the polynomial \( f(x) \) if and only if \( f(a) = 0 \).

Suppose you wanted to find the factors of \( x^3 - 3x^2 - 6x + 8 \). One approach is to graph the related function, \( f(x) = x^3 - 3x^2 - 6x + 8 \). From the graph at the right, you can see that the graph of \( f(x) \) crosses the \( x \)-axis at \(-2, 1 \), and \( 4 \). These are the zeros of the function. Using these zeros and the Zero Product Property, we can express the polynomial in factored form.

\[
f(x) = (x - (-2))(x - 1)(x - 4) = (x + 2)(x - 1)(x - 4)
\]

What happens when a factor repeats?

**Multiplicity of Roots (or Zeros):** We just saw that the real roots (zeros) of a polynomial correspond with the \( x \)-intercepts of the polynomial graph. In some situations, the graph will "cross" the \( x \)-axis at these points. In other situations, the graph may simply "touch" (be tangent to) the \( x \)-axis at these points. Let's see if we can determine, before we draw the graph, whether it will "cross" the \( x \)-axis at each root, or simply "touch" (be tangent to) the \( x \)-axis at each root.

Consider the example at the right. The polynomial is of degree two, so there will be two roots (zeros). The factor of \( (x + 3) \) is repeated twice, and can also be written as \( (x + 3)^2 \).

**Definition:** The number of times a factor appears in a polynomial is referred to as its **multiplicity**.

In the example at the right, the factor \( (x + 3) \) has a multiplicity of 2, since it appears twice. It creates a "repeated root".

**Multiplicity EVEN:** When the multiplicity (the number of times a factor repeats) is an even number, the graph will just "touch" (be tangent to) the \( x \)-axis at that point.

Why? "EVEN" multiplicities are factors that occur an even number of times, and form squares. Since squares are always positive, the graph near the root (zero) will not change signs from positive (above the \( x \)-axis) to negative (below the \( x \)-axis), or vice versa. The graph will "touch", or "bounce off", the \( x \)-axis at the root (zero) but remain on the same side of the \( x \)-axis.

**Example degree 2:**

\[
P(x) = x^2 + 6x + 9
\]

\[ (x + 3)(x + 3) = 0 \]

repeated factor: \( (x + 3)^2 \)

Set each factor = 0.

\[
\begin{align*}
  x + 3 &= 0 \\
  x &= -3 \\
  x + 3 &= 0 \\
  x &= -3
\end{align*}
\]

Roots: \( x = -3 \)

"just touching" the \( x \)-axis may also be referred to as "bouncing off" the \( x \)-axis.
Comprehension questions: part 2 (IDing factors of polynomial functions)

Answer each of the following questions on a separate sheet of paper.

1. Use the graph below, which shows the function $k(x)$, to answer each question:

   a) What is the degree of $k(x)$? Explain how you know.
   b) Is the leading coefficient positive or negative? Explain how you know.
   c) Marcos observes that $k$ has $x$-intercepts at $x = -5$, $x = 0$, and $x = 3$ and writes the equation $k(x) = a(x - 5)(x)(x + 3)$, where $a$ is the unknown leading coefficient. Explain why Marcos’s equation is incorrect. Then write a correct equation for $k$.

2. Use the graph of $m(x)$, below, to answer each question:

   a) Which root(s) have a multiplicity? Explain how you know.
   b) Write the equation for $m(x)$ in factored form.
   c) $m(x)$ contains the point $(-2, -1)$. What is the value of $a$?

3. A polynomial function $n(x)$ is given by $n(x) = -2(x + 5)^2(x - 1)(x - 4)$. Without using a calculator, (a) identify the degree of the function, (b) describe the end behavior of the function, and (c) sketch a graph of the function on the $(x, y)$ coordinate plane.
III. Polynomial long division

We have seen simple division of polynomials in Algebra 1, such as:

\[
\frac{4x^2 - 8}{4} = \frac{A(x - 8)}{A} = x - 8 \\
\frac{6x^3 + 9x^2}{3x} = \frac{6x^2 + 9x^2}{3x} = 2x^2 + 3x
\]

Dividing a factorable polynomial by a binomial:

\[
\frac{2x^2 + 7x + 6}{x^2 - 4} = \frac{(2x + 3)(x + 2)}{(x + 2)(x - 2)} = \frac{2x + 3}{x - 2}
\]

It is now time to expand our division skills to allow us to conquer more situations.

Long Division

Our first new strategy is LONG DIVISION. Yes, it is the same idea that you learned in elementary school. Let's see how our new process resembles what you already know how to do.

**Numerical Long Division Process:**

\[
\begin{array}{c|ccccc}
\text{(divisor)} & 2x^2 & +7x & +6 & \\
\hline 
3 & 2x^2 & +6x & +3 & \\
\downarrow \\
27 & -6x & -18 & & \\
\downarrow \\
24 & & -18 & & \\
\downarrow \\
24 & & & -27 & & \\
\downarrow \\
-27 & & & & +1 & \\
\downarrow \\
1 & & & & & \\
\end{array}
\]

**Algebraic Long Division Process:**

\[
\begin{array}{c|ccccc}
\text{x + 2} & 2x^2 & +7x & +6 & \\
\hline 
\text{multiply} & 2x & +3 & & & \\
\downarrow \\
2x^2 & +4x & +3 & & & \\
\downarrow \\
26 & -3x & -6 & & & \\
\downarrow \\
24 & -3x & -6 & & & \\
\downarrow \\
24 & & & & -27 & & \\
\downarrow \\
-27 & & & & +1 & & \\
\downarrow \\
1 & & & & & \\
\end{array}
\]

There is no remainder to be listed in this problem.

**Division Algorithm:**

\[
\frac{\text{Dividend}}{\text{Divisor}} = \text{Quotient} + \frac{\text{Remainder}}{\text{Divisor}}
\]

More formally stated, the division algorithm algebraically states that:

If \( f(x) \) and \( d(x) \neq 0 \) are polynomials such that the degree of \( d(x) \) \( \leq \) the degree of \( f(x) \), then there exists unique polynomials \( q(x) \) and \( r(x) \) such that

\[
\frac{f(x)}{d(x)} = q(x) + \frac{r(x)}{d(x)}
\]

and the degree of \( r(x) \) < the degree of \( d(x) \).

If \( r(x) = 0 \), then \( d(x) \) divides evenly into \( f(x) \), making \( d(x) \) a factor of \( f(x) \).

This algorithm is simply saying that when the two polynomials are divided \((f(x) \div d(x))\), the solution will be the quotient, \( q(x) \), plus a remainder expressed as the remainder over the divisor, \( r(x)/d(x) \).
Let's examine algebraic long division in a variety of situations. We will be assuming that the divisors in these examples are not zero (i.e., in Example 1, assume \(x - 3 \neq 0\)).

### Example 1:

**Process:**

1. Start the division by asking, "What term multiplied times \(x\) will give \(2x^3\)?" In this problem, the answer is \(2x^2\).

2. Multiply your answer to this question times the divisor \((x - 3)\), lining up the similar terms. Then subtract (being careful to change the signs).

3. Bring down the next available term in the dividend.

4. Repeat this process (from step 1 through step 3) until all terms from the dividend have been used.

5. If there is a remainder, place the remainder over the divisor and add it to your quotient answer. (This is the same manner of expressing the remainder that you saw in elementary long division.)

<table>
<thead>
<tr>
<th>Example 1:</th>
<th>Divide: ((2x^3 + 4x^2 + 5x - 1)) by ((x - 3))</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Process:</strong></td>
<td><strong>Result:</strong></td>
</tr>
<tr>
<td>1. Start the division by asking, &quot;What term multiplied times (x) will give (2x^3)?&quot; In this problem, the answer is (2x^2).</td>
<td>(2x^2 + 10x + 35)</td>
</tr>
<tr>
<td>2. Multiply your answer to this question times the divisor ((x - 3)), lining up the similar terms. Then subtract (being careful to change the signs).</td>
<td>(\frac{104}{x - 3})</td>
</tr>
<tr>
<td>3. Bring down the next available term in the dividend.</td>
<td>(x - 3)</td>
</tr>
<tr>
<td>4. Repeat this process (from step 1 through step 3) until all terms from the dividend have been used.</td>
<td><strong>Remainder over divisor.</strong></td>
</tr>
<tr>
<td>5. If there is a remainder, place the remainder over the divisor and add it to your quotient answer. (This is the same manner of expressing the remainder that you saw in elementary long division.)</td>
<td><strong>Remainder over divisor.</strong></td>
</tr>
</tbody>
</table>

### Example 2:

**Beware:** Be sure that the polynomial is in **descending order** (by powers).

If not, re-write the polynomial into descending order before beginning the division.

<table>
<thead>
<tr>
<th>Example 2:</th>
<th>Divide: ((3x^2 + x^3 - 2x + 6)) by ((x - 1))</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Process:</strong></td>
<td><strong>Result:</strong></td>
</tr>
<tr>
<td>Be sure that the polynomial is in <strong>descending order</strong> (by powers).</td>
<td>((x^3 + 3x^2 - 2x + 6)) by ((x - 1))</td>
</tr>
<tr>
<td>If not, re-write the polynomial into descending order before beginning the division.</td>
<td>(x^3 + 3x^2 - 2x + 6)</td>
</tr>
</tbody>
</table>
In this case, there is no remainder and so we say that $a - 2$ is a factor of $a^3 - 8$.
In this case, we can rewrite the problem as a multiplication problem: $a^3 - 8 = (a - 2)(a^2 + 2a + 4)$!

**Comprehension questions: part 3 (polynomial long division)**

Answer each of the following questions on a separate sheet of paper.

1. The process in examples #1 – 3, above, is called polynomial long division. How is polynomial long division similar to long division with whole numbers? How is it different?

2. Use the long division work shown below. How could the problem be written as the sum of a quotient and a remainder over a divisor? Fill in the boxes to complete your answer.

3. Hector is given $P(x) = x^4 - x^2 - 2$ and asked to determine if $x^2 - 2$ is a factor of $P(x)$. He completes the work below:

   Is Hector correct? If not, explain his mistake. If so, explain how he could check his work.

4. What is the remainder when $x^3 - 57x + 56$ is divided by $x - 7$?
IV. Factoring polynomials by grouping

There are many different applications of factoring by grouping. We will be examining factoring by grouping as it applies to trinomials of the form $ax^2 + bx + c$.

Notice: $(x + 4)(x + 6) = x^2 + 4x + 6x + 24 = x^2 + 10x + 24$

Whenever you multiply two binomials, you create two "middle" terms, which in many cases are "like" terms. It is the creation of these "middle" terms which becomes the focus of the process of factoring by grouping for a trinomial. This method is often called "split the middle" since it endeavors to create two "middle" terms which will make the factoring process easier.

**Example 1:**

| **Factor: $8x^2 + 26x + 15$** |  
|---|---|
| **1.** Always check for any common factors before you begin. It will make finding the solution easier. | There are no common factors in this trinomial. |
| **2.** Find $a \cdot c$ (referred to as the Master Product). | $a \cdot c = 8 \cdot 15 = 120$ |
| **3.** Find two new factors of $a \cdot c$ (120) that add up to $b$ (+26). Let the two factors be $m$ and $n$. Then $m \cdot n = 120$ and $m + n = 26$. The new factors are $m = 20$ and $n = 6$. | New factors: $20 \cdot 6 = 120$
$20 + 6 = 26$ |
| **4.** Split the middle term into two terms using the sum of the two new factors, including the proper signs. | $8x^2 + 20x + 6x + 15$ |
| **5.** Group the four terms to form two pairs. Be careful of the signs. | $(8x^2 + 20x) + (6x + 15)$ |
| **6.** Factor each pair by finding the common factors. | $4x(2x + 5) + 3(2x + 5)$ |
| **7.** Factor out the common (shared) binomial. | $(2x + 5)(4x + 3)$ |

**Example 2:**

| **Factor: $5x^3 + 15x^2 - 10x - 30$** |  
|---|---|
| **Factor out the GCF, greatest common factor, of 5.** | $5[x^3 + 3x^2 - 2x - 6]$ |
| **Group the first two terms. Group the second two terms after factoring out a -1.** | $5[(x^3 + 3x^2) - 1(2x + 6)]$ |
| **Factor out $x^2$ from the first two grouped terms. Factor out -2 from the second two grouped terms.** | $5[x^2(x + 3) - 1\cdot 2(x + 3)]$
$5[x^2(x + 3) - 2(x + 3)]$ |
| **Notice that the parentheses in both groups contain the same value of $(x + 3)$. Factor out the common parentheses.** | $5(x + 3)(x^2 - 2)$ |

**HINT**

It may be the case in the second grouping, that you may be able to factor out either a positive value or a negative value. To determine which is needed in the problem, look at the signs before the second and fourth terms. If the two signs are the same (either both positive or both negative), factor out a positive value. If the two signs are different, factor out a negative value.
Example 3:

<table>
<thead>
<tr>
<th>Factor: $5x^3 + 7x^2 - 20x - 28$</th>
</tr>
</thead>
<tbody>
<tr>
<td>There is no GCF, so group the terms. Be careful of the negative sign in front of the second grouping.</td>
</tr>
<tr>
<td>$(5x^3 + 7x^2) - (20x + 28)$</td>
</tr>
<tr>
<td>Factor the groupings.</td>
</tr>
<tr>
<td>$x^2(5x + 7) - 4(5x + 7)$</td>
</tr>
<tr>
<td>Factor out the common parentheses of $(5x + 7)$.</td>
</tr>
<tr>
<td>$(5x + 7)(x^2 - 4)$</td>
</tr>
<tr>
<td>Be careful! You are not finished. The $(x^2 - 4)$ can be factored further!</td>
</tr>
<tr>
<td>$(5x + 7)(x - 2)(x + 2)$</td>
</tr>
</tbody>
</table>

Example 4:

<table>
<thead>
<tr>
<th>Factor: $a^4b - 2a^3 + a^4 - 2a^3b$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Factor out the GCF, greatest common factor, of $a^2$.</td>
</tr>
<tr>
<td>$a^3(ab - 2 + a - 2b)$</td>
</tr>
<tr>
<td>Look ahead! Re-group to create the same value in the grouped parentheses.</td>
</tr>
<tr>
<td>$a^3(ab - 2b + a - 2)$</td>
</tr>
<tr>
<td>$a^3[(ab - 2b) + (a - 2)]$</td>
</tr>
<tr>
<td>Factor the groupings.</td>
</tr>
<tr>
<td>$a^3[b(a - 2) + 1(a - 2)]$</td>
</tr>
<tr>
<td>Factor out the common parentheses.</td>
</tr>
<tr>
<td>$a^3[(a - 2)(b + 1)]$</td>
</tr>
<tr>
<td>Final answer.</td>
</tr>
<tr>
<td>$a^3(a - 2)(b + 1)$</td>
</tr>
</tbody>
</table>

Comprehension questions: part 4 (factoring polynomials by grouping)

Answer each of the following questions on a separate sheet of paper.

1. Use the problem below, which shows the process for factoring $2x^2 + 5x - 3$ by grouping, to answer each question.
   \[2x^2 + 6x - x - 3 = 2x(x+3) - (x+3) = (x+3)(2x-1)\]
   a) How did the student know to split the second term $(5x)$ into $6x - x$?
   b) Explain why the negative sign is placed outside of the parenthesis in the second step.

2. Factor completely:
   \[a^3 - 4a^2 + 3a - 12\]

3. Factor the trinomial given by $12z^2 + 22z + 8$.

4. Factor completely:
   \[x^5 + x^4 - x^3 - x^2\]
V. Graphs and transformations of radical functions

An important note:
Remember that the domain of a function is the set of all possible x-values that satisfy the function, while the range of a function is the set of all possible y-values that satisfy the function.

The domain and range of a function can be written in three ways:

- **Verbally:** for the parent function \( y = \sqrt{x} \), the domain is all values of \( x \) that are greater than or equal to 0.
- **Interval notation:** for the parent function \( y = \sqrt{x} \), the domain is \( x \geq 0 \) (note the similarity to the verbal representation; we read this inequality as “all values of \( x \) that are greater than or equal to zero”)
- **Set notation:** for the parent function \( y = \sqrt{x} \), the domain is \( [0, \infty) \); note that the bracket symbol means 0 is included in the domain and the parenthesis means that \( \infty \) is not included in the domain.
Comprehension questions: part 5 (transformations of radical functions)

Answer each of the following questions on a separate sheet of paper.

1. Use the functions $f(x) = \sqrt{x}$ and $g(x) = \sqrt{x} - 5$. What transformation(s) is/are needed to map $f(x)$ onto $g(x)$?

2. Use the functions $f(x) = \sqrt{x}$ and $h(x) = \sqrt{x} + 4$. What transformation(s) is/are needed to map $f(x)$ onto $g(x)$?

3. Use the functions $f(x) = \sqrt{x}$ and $j(x) = -\frac{1}{2} \sqrt{x}$. What transformation(s) is/are needed to map $f(x)$ onto $g(x)$?

4. **Without graphing**, identify the transformation(s) needed to create $k(x) = 4\sqrt{x + 1} - 2$ from the parent function $f(x) = \sqrt{x}$. Explain your thinking.

5. Graph the functions $m(x) = -\sqrt{x}$ and $n(x) = \sqrt{-x}$ on the same coordinate plane. How does the graph of $m$ differ from the graph of $n$? Explain why this type of transformation does not exist for quadratic functions.

6. Match each function below with its' graph.

   (i) $y = \sqrt{x} - 2$   (ii) $y = -\sqrt{x} + 2$   (iii) $y = -\sqrt{x + 2}$   (iv) $y = -\sqrt{(x - 2)}$

   ![Graphs](image)

7. Identify the domain and range of the function $y = \sqrt{x} - 4$.

8. Sketch a graph of the function $y = 3\sqrt{x + 2}$, showing key features.
VI. Solving radical equations

**Definition:** A radical equation is an equation that has a variable in a radicand (or a variable with a rational exponent).

On this page, all radicals and expressions with rational exponents represent real numbers.

The "radical" in a radical equation may be of any root value: square root, cube root, fourth root, etc. This page will concentrate on working with "square root" equations.

**Hint:** Remember that a radical equation has the variable "under" the radical, not simply a numerical radical value within the equation.

\[
\sqrt{x + 12} = 48 \quad \text{is a radical equation}
\]

\[
x + \sqrt{2} = 8 \quad \text{is NOT a radical equation}
\]

The solution to a radical equation is a real number which, when substituted for the variable, yields a true equation. Radical equations with square roots can be solved by "squaring", with cube roots by "cubing", and so on. This solution process is an application of the principle:

If \( a \) and \( b \) are real numbers, \( n \) is a positive integer, and \( a = b \), then \( a^n = b^n \).

The solutions, however, get significantly "messier" to find as the indexes increase.

**To solve radical equations:**

1. Isolate the radical (or one of the radicals) to one side of the equal sign.
2. If the radical is a square root, square each side of the equations. If the radical is not a square root, raise each side to a power equal to the index of the root.
3. Solve the resulting equation.
4. Check your answer(s) to avoid extraneous roots.

**Beware:**

There may be a problem with these types of equations as some of the "perceived" solutions may not actually work when plugged back into the original equations. This problem arises from the fact that the converse of the statement: If \( a \) and \( b \) are real numbers, \( n \) is a positive integer, and \( a = b \), then \( a^n = b^n \), is true when \( n \) is odd, but not necessarily true when \( n \) is even.

The process of squaring the sides of an equation creates a "derived" equation which may not be equivalent to the original radical equation. Consequently, solving this new derived equation may create solutions that are not related to the original equation. These "extra" roots that are not true solutions of the original radical equation are called extraneous roots and are rejected as answers.

CHECKING will be essential to finding correct solutions (and catching these pesky extraneous roots).
Example 1: Example where the answer checks!

\[ \sqrt{x - 3} = 4 \]

\begin{align*}
\left( \sqrt{x - 3} \right)^2 &= (4)^2 \\
x - 3 &= 16 \\
x &= 19
\end{align*}

CHECK: \[ \sqrt{19 - 3} = \sqrt{16} = 4 \] (check)

\[ \text{ANSWER: } x = 19 \]

Example 2: Example where the answer does NOT check!

\[ \sqrt{2x - 1} + 5 = 2 \]

\begin{align*}
\sqrt{2x - 1} &= -3 \\
\left( \sqrt{2x - 1} \right)^2 &= (-3)^2 \\
2x - 1 &= 9 \\
2x &= 10 \\
x &= 5
\end{align*}

CHECK: \[ \sqrt{2(5) - 1} + 5 = \sqrt{9} + 5 = 8 \neq 2 \]

\[ \text{(does NOT check)} \]

\[ \text{ANSWER: no solution} \]

Example 3: Example where ONLY ONE answer checks!

Solve: \[ \sqrt{-2x + 30} = x - 3 \]

\begin{align*}
\left( \sqrt{-2x + 30} \right)^2 &= (x - 3)^2 \\
-2x + 30 &= x^2 - 6x + 9 \\
0 &= x^2 - 4x - 21 \\
0 &= (x - 7)(x + 3) \\
x &= 7; \ x = -3
\end{align*}

CHECK: \[ \sqrt{-2(7) + 30} = 7 - 3 \]

\[ 4 = 4 \] (check)

CHECK: \[ \sqrt{-2(-3) + 30} = -3 - 3 \]

\[ 6 \neq -6 \] (does not check)

\[ \text{ANSWER: } x = 7 \]
Comprehension questions: part 6 (solving radical equations)

Answer each of the following questions on a separate sheet of paper.

1. Solve each equation below. Check that your answer(s) satisfy the original equation.
   a) $2\sqrt{x} - 8 = 0$
   b) $\sqrt{3x - 7} + 2 = 5$
   c) $\sqrt{2 - x} = x + 4$

2. A student is solving the equation $x = \sqrt{18 - 7x}$, shown below. Identify and correct their error.

3. Find all values of $x$ that make the equation true:
   \[ \sqrt{3 - x} = 3 - \sqrt{x + 2} \]
VII. Radical equations in real-world contexts

Answer each of the following questions on a separate sheet of paper. (Note that there are no examples provided for this section; you should use the resources from earlier sections to help you apply your skills)

1. The equation \( V = 20\sqrt{C + 273} \) relates the speed of sound, \( V \), in meters per second, to the air temperature, \( C \), in degrees Celsius. Marie and Nick have been asked to find the temperature that will make the speed of sound 320 meters per second, and write two different equations:

<table>
<thead>
<tr>
<th>Marie</th>
<th>Nick</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 320 = 20\sqrt{C + 273} )</td>
<td>( V = 20\sqrt{320 + 273} )</td>
</tr>
</tbody>
</table>

Whose equation is correct? Explain your answer.

2. The annual banana consumption \( y \) (in pounds per person) in the United States can be modeled by the function \( y = \sqrt{18x + 272} \), where \( x \) is the number of years since 1970. Which equation at right could be used to find the year in which about 20 pounds of bananas were consumed per person?

(A) \( y = \sqrt{18(20) + 272} \)
(B) \( y = \sqrt{18(1970) + 272} \)
(C) \( 20 = \sqrt{18x + 272} \)
(D) \( 1970 = \sqrt{18x + 272} \)
(E) \( 1970 - 20 = \sqrt{18x + 272} \)

3. Use your equation from exercise #2, above, to predict the year in which about 20 pounds of bananas were consumed per person.

4. Bobby, a forensic scientist, is called to the scene of a crime first thing in the morning. Two cars crashed into a store last night, destroying it. Both cars are still at the scene of the crime, and the police are sure that at least one of the two cars was speeding. However, because of the extensive damage, they aren't sure which one - both drivers are accusing the other driver of being the one going too fast.

As Bobby analyzes the two sets of skid marks left by the cars, he recalls that the speed of a car in miles per hour \( s \) can be found by using the formula \( S = 15.9\sqrt{d} \) where \( d \) is the length of skid marks left by the car’s tires on the road, in meters.

a) If the first car, a 1999 Toyota Corolla, left skid marks that are 4 meters long and the second car, a 2003 Mazda 3 hatchback, left skid marks that are 12 meters long, how fast was each car traveling?

b) The speed limit in the vicinity of the crime was 40 miles per hour. What is the length of skid marks that would correspond to this speed?

5. The power \( P \) in watts that a circular solar cell produces and the radius of the cell in centimeters are related by the equation \( r = \frac{P}{0.02\pi} \). For each question below, show the algebraic work leading to your answer and round decimals to the nearest hundredth.

a) About how much power is produced by a cell with a radius of 10 cm?

b) What is the radius of the cell that produces 50 watts of power?
VIII. Composite functions

The term "composition of functions" (or "composite function") refers to the combining together of two or more functions in a manner where the output from one function becomes the input for the next function.

Mathematically speaking, the range (the y-values) of one function becomes the domain (the x-values) of the next function.

The notation used for composition is:

\[(f \circ g)(x) = f(g(x))\]

and is read "f composed with g of x" or "f of g of x".

Notice how the letters stay in the same order in each expression for the composition. The letters \(f(g(x))\) tell you to start with the function \(g\) (always start with the function in the innermost parentheses).

If you become confused as to where to start when you see the notation \((f \circ g)(x)\), start by rewriting the notation to be \(f(g(x))\). Then you can see that you need to start with the innermost parentheses, which in this case will be function \(g\).

Reminder: When dealing with composition ...

Always start with the function on the right! \(f(g(x))\)

Algebraic Examples:

1. Given \(f(x) = 5x + 1\) and \(g(x) = 3x - 2\). Express \((f \circ g)(x)\) in terms of \(x\).

Solution:
Remember \((f \circ g)(x) = f(g(x))\).
\[
\begin{align*}
(f \circ g)(x) &= f(3x - 2) \\
&= f(3x) + 1 \\
&= 5(3x - 2) + 1 \\
&= 15x - 10 + 1 \\
&= 15x - 9
\end{align*}
\]

Substitute \(3x - 2\) in place of \(x\) in the function \(f(x)\).
Simplify using the distributive property.

2. Given \(f(x) = 3x^2 + 4x + 7\) and \(g(x) = x + 1\). Express a) \((f \circ g)(x)\) and b) \((g \circ f)(x)\).

Solution:

a) \((f \circ g)(x) = f(g(x)) = f(x + 1)\)
\[
\begin{align*}
(f \circ g)(x) &= 3(x + 1)^2 + 4(x + 1) + 7 \\
&= 3(x^2 + 2x + 1) + 4x + 4 + 7 \\
&= 3x^2 + 6x + 3 + 4x + 11 \\
&= 3x^2 + 10x + 14
\end{align*}
\]
Substitute \(x + 1\) in place of \(x\) in the function \(f(x)\).
Simplify – remember that \((x + 1)^2 = (x + 1)(x + 1)\)
\[
= x^2 + 1x + 1x + 1
\]

b) \((g \circ f)(x) = g(f(x)) = g(3x^2 + 4x + 7)\)
\[
\begin{align*}
(g \circ f)(x) &= 3x^2 + 4x + 7 + 1 \\
&= 3x^2 + 4x + 8
\end{align*}
\]
Substitute \(3x^2 + 4x + 7\) in place of \(x\) in the function \(g(x)\).
No distribution necessary – just add like terms to simplify!
3. Given \( f(x) = x^2 - 1 \) and \( g(x) = 3x + 2 \).
Express a) \( (f \circ g)(2) \), b) \( (g \circ f)(4) \), c) \( g(g(-1)) \)

Solution:

\[
\begin{align*}
\text{a)} & \quad (f \circ g)(2) = f(g(2)) \\
& = f(3(2) + 2) \\
& = f(8) \\
& = 8^2 - 1 = 63 \\
\text{b)} & \quad (g \circ f)(4) = g(f(4)) \\
& = g(4^2 - 1) \\
& = g(15) \\
& = 3(15) + 2 = 47 \\
\text{c)} & \quad g(g(-1)) = g(3(-1) + 2) \\
& = g(-3 + 2) \\
& = g(1) \\
& = 3(-1) + 2 = -1
\end{align*}
\]

Note that you can also perform this as two different steps:
\[
(f \circ g)(2) = f(g(2)) \\
g(2) = 3(2) + 2 = 6 + 2 = 8 \\
f(8) = 8^2 - 1 = 64 - 1 = 63 \\
\text{Therefore, } f(8) = 63
\]

**Comprehension questions: part 8 (composite functions)**

Answer each of the following questions on a separate sheet of paper.

1. Given \( p(x) = x^2 + 4x + 1 \) and \( q(x) = -2x \). Which of the following is equivalent to \( q(p(x)) \)?

   \[\begin{align*}
(A) & \quad -2x^4 - 8x^3 - 2x^2 \\
(B) & \quad -2x^3 - 8x^2 - 2x \\
(C) & \quad -2x^2 - 8x - 2 \\
(D) & \quad 4x^3 - 8x^2 + x \\
(E) & \quad 4x^2 - 8x + 1
\end{align*}\]

2. Use the chart below. Which of the following is the value of \( g(f(4)) \)?

   \[
\begin{array}{c|c|c|c|c|c|}
 x & -6 & 0 & 2 & 4 \\
 \hline
 f(x) & 4 & -6 & 0 & 2 \\
 g(x) & 2 & 4 & -6 & 0 \\
 h(x) & 0 & 2 & 4 & -6 \\
 k(x) & 1 & 4 & 0 & 3 \\
\end{array}
\]

   \[\begin{align*}
(A) & \quad -6 \\
(B) & \quad 0 \\
(C) & \quad 1 \\
(D) & \quad 2 \\
(E) & \quad 3 \\
(F) & \quad 4
\end{align*}\]

3. Let \( t(x) = x^2 + 5x - 1 \), \( v(x) = \frac{18x}{2x^2 + 7} \), \( w(x) = \frac{3}{2}x + 3 \), and \( z(x) = x - 1 \).
   a) Find \( w(v(1)) \).
   b) Find \( t(z(x)) \).
   c) Write an expression for \( (z \circ w)(x) \).

4. The number \( N \) of bacteria in a refrigerated food is given by \( N(T) = 20T^2 - 80T + 500 \) for \( 2 \leq T \leq 14 \) where \( T \) is the temperature of the food in degrees Celsius. Once the food is removed from refrigeration, the temperature of the food is given by \( T(t) = 4t + 2 \) for \( 0 \leq t \leq 3 \) where \( t \) is the time in hours.
   Calculate \( N(T(2)) \). Interpret the meaning of this value in the context of the problem.
IX. Solving exponential equations using common bases

**Exponential equations** are equations in which variables occur as exponents. The following property is useful for solving exponential equations.

### Key Concept

**Property of Equality for Exponential Functions**

- **Symbols**
  - If \( b \) is a positive number other than 1, then \( b^x = b^y \)
  - If and only if \( x = y \).

- **Example**
  - If \( 2^x = 2^8 \), then \( x = 8 \).

#### Example 1

**Solve Exponential Equations**

Solve each equation.

a. \( 3^{2n + 1} = 81 \)

Original equation:

\[ 3^{2n + 1} = 81 \]

Rewrite 81 as \( 3^4 \) so each side has the same base:

\[ 3^4 = 81 \]

Property of Equality for Exponential Functions:

\[ 2n + 1 = 4 \]

Subtract 1 from each side:

\[ 2n = 3 \]

\[ n = \frac{3}{2} \]

Divide each side by 2.

The solution is \( n = \frac{3}{2} \).

**CHECK**

Original equation:

\[ 3^{2n + 1} = 81 \]

Substitute \( n = \frac{3}{2} \) for \( n \):

\[ 3^{2n + 1} = 3^{\frac{3}{2} + 1} = 3^{\frac{5}{2}} \]

Simplify:

\[ 3^{\frac{5}{2}} \]

\[ 81 = 81 \] (Simplify)

b. \( 4^{2x} = 8^{x - 1} \)

Original equation:

\[ 4^{2x} = 8^{x - 1} \]

Rewrite each side with a base of 2:

\[ (2^2)^{2x} = (2^3)^{x - 1} \]

Power of a Power:

\[ 2^{4x} = 2^{3(x - 1)} \]

Property of Equality for Exponential Functions:

\[ 4x = 3(x - 1) \]

Distributive Property:

\[ 4x = 3x - 3 \]

Subtract 3x from each side:

\[ x = -3 \]

The solution is \( x = -3 \).

#### Example 2

**Solve Exponential Inequalities**

Solve \( 4^{3p - 1} > \frac{1}{256} \).

Original inequality:

\[ 4^{3p - 1} > \frac{1}{256} \]

Rewrite \( \frac{1}{256} \) as \( \frac{1}{4^4} \) or \( 4^{-4} \) so each side has the same base:

\[ 3p - 1 > -4 \]

Property of Inequality for Exponential Functions:

\[ 3p > -3 \]

Add 1 to each side:

\[ p > -1 \]

Divide each side by 3.

The solution set is \( p > -1 \).

**CHECK**

Test a value of \( p \) greater than \(-1\); for example, \( p = 0 \).

Original inequality:

\[ 4^{3p - 1} > \frac{1}{256} \]

Replace \( p \) with 0:

\[ 4^{3(0) - 1} > \frac{1}{256} \]

Simplify:

\[ 4^{-1} > \frac{1}{256} \]

\[ \frac{1}{4} > \frac{1}{256} \] (Simplify)

\[ a^{-1} = \frac{1}{a} \]
Comprehension questions: part 9 (solving exponential equations using a common base)

Answer each of the following questions on a separate sheet of paper.

1. If $4^{a^2+4a} = 4^{-3}$, which of the following is the value of $a$?
   (A) $a = 1$ or $a = 3$
   (B) $a = -1$ or $a = 3$
   (C) $a = -1$ or $a = -3$
   (D) $a = 1$ or $a = -3$

2. Maria is trying to solve the equation $7^{y-3} = 1$.
   a) Explain what Maria’s first step should be to solve the equation. Why is this step necessary?
   b) Use your answer from part (a) to rewrite the equation and solve for $y$.

3. Solve each equation below, showing all relevant work.
   Check that your answer(s) satisfy the original equation.
   a. $9^{3x-2} = 9^{2x+1}$
   b. $27^w < 3^{2w-1}$
   c. $8^{4x-3} = 32^{2x+1}$
   d. $5^{m^2+1} = \left(\frac{1}{25}\right)^m$

4. Use the equation $2^x = 28$. Explain why this equation cannot be solved by using the common base strategy.
X. Solving exponential equations using logarithms

SOLVE LOGARITHMIC EQUATIONS You can use the properties of logarithms to solve equations.

Example 1: Solve Logarithmic Equations Using Exponentiation

EARTHQUAKES The amount of energy $E$, in ergs, that an earthquake releases is related to its Richter scale magnitude $M$ by the equation $\log E = 11.8 + 1.5M$. The Chilean earthquake of 1960 measured 8.5 on the Richter scale. How much energy was released?

\[
\begin{align*}
\log E &= 11.8 + 1.5M \\
\log E &= 24.55 \\
10^{\log E} &= 10^{24.55} \\
E &= 10^{24.55} \\
E &\approx 3.55 \times 10^{24}
\end{align*}
\]

Remember that if their bases are the same, exponential and logarithmic functions “undo” each other, so $10^{\log E} = E$.

The amount of energy released by this earthquake was about $3.55 \times 10^{24}$ ergs.

Example 2: Solve Exponential Equations Using Logarithms

Solve $3^x = 11$.

\[
\begin{align*}
3^x &= 11 \\
\log 3^x &= \log 11 \\
x \log 3 &= \log 11 \\
x &= \frac{\log 11}{\log 3} \\
x &\approx 2.1828
\end{align*}
\]

CHECK You can check this answer using a calculator or by using estimation. Since $3^2 = 9$ and $3^3 = 27$, the value of $x$ is between 2 and 3. In addition, the value of $x$ should be closer to 2 than 3, since 11 is closer to 9 than 27. Thus, 2.1828 is a reasonable solution.

EQUATIONS AND INEQUALITIES WITH $e$ AND $\ln$ Equations and inequalities involving base $e$ are easier to solve using natural logarithms than using common logarithms. All of the properties of logarithms that you have learned apply to natural logarithms as well.

Example 3: Solve Base $e$ Equations

Solve $5e^{-x} - 7 = 2$.

\[
\begin{align*}
5e^{-x} - 7 &= 2 \\
5e^{-x} &= 9 \\
e^{-x} &= \frac{9}{5} \\
\ln e^{-x} &= \ln \frac{9}{5} \\
-x &= \ln \frac{9}{5} \\
x &= -\ln \frac{9}{5} \\
x &\approx -0.5878
\end{align*}
\]

CHECK You can check this value by substituting $-0.5878$ into the original equation or by finding the intersection of the graphs of $y = 5e^{-x} - 7$ and $y = 2$.

Remember that the natural logarithm (ln) is a logarithm with a base of $e$; the natural logarithm cancels with $e$ on this side of the equation!
Comprehension questions: part 10 (solving exponential equations using logarithms)

Answer each of the following questions on a separate sheet of paper.

1. If $87e^{0.3x} = 5918$, then the value of $x$ is approximately...
   
   (A) 0.583  
   (B) 1.945  
   (C) 4.220  
   (D) 14.066

2. Which of the following is the solution to $2^{x+3} = 6$?
   
   (A) $\log_2 3 - 3$  
   (B) 0  
   (C) $\frac{\log 6}{\log 2} + 3$  
   (D) $\frac{\log 2}{\log 6} - 3$

3. Solve the equation. Give both an exact solution as well as an approximate solution rounded to 3 decimal places.
   
   a) $15 = 3^{2x+1}$
   
   b) $e^x = 43$
   
   c) $4^{x+2} - 2 > 12$

4. Carlos took a dose of prescription medication. The amount of medicine (in milligrams) remaining in his bloodstream after $t$ hours is given by $M(t) = 20e^{-0.8t}$.
   
   a) How much medication is remaining in Carlos' bloodstream 2 hours after he took the dose? Show all work leading to your answer, include correct units, and round your answer to the nearest thousandth. (HINT: the $e^x$ key on your calculator is located above the "LN" key; you can also use Google's calculator or on your phone!)
   
   b) When Carlos has less than 1 mg of medication remaining in his bloodstream, he needs to take a second dose. About how long should Carlos wait to take the second dose of medication?